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ABSTRACT

This paper considers a sequence of piecewise linear orthogonal functions (U_i) which is complete in L_2. Explicit expressions for the U_i are given.

Any continuous function can be expanded in a series of constant multiples of functions U_i under the sense of uniform convergence by group. While the partial sum of the corresponding expression for a continuous function may fail to converge uniformly, a certain subsequence does.

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SIGNIFICANCE AND EXPLANATION

Fourier analysis is the basis for development in many areas. Haar function and Walsh function as a significant example of non-sinusoidal functions which form complete sets provided an effectual tool in Fourier analysis for some subjects. From the point of view of approximation theory and its applications, a set of piecewise smooth orthonormal functions should have some advantages. Haar and Walsh functions are piecewise constants. Franklin considered a polygonal system. But it is not convenient to use since its description is implicit.

This paper considers a set of discontinuous piecewise linear functions, presents their explicit representations, proves the orthonormality and completeness in Hilbert space $L_2[0,1]$. The corresponding Fourier series for any continuous function is uniformly convergent by group.

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A SEQUENCE OF PIECEWISE ORTHOGONAL POLYNOMIALS (I) (LINEAR CASE)

Y. Y. Feng* and D. X. Qi**

1. Introduction

It is very important to study orthogonal functions. A number of function sets are known which are found to be orthogonal and hence can be used for series representation. Fourier series, a system of sine and cosine functions, is the basis for development in many areas.

Additionally certain polynomials can be made orthogonal. These orthogonal polynomials form a series, $\varphi_n(x)$ $(n=0,1,2,\ldots)$, where n is the degree of the polynomial. This class contains many special functions commonly encountered in practical applications, e.g. Chebyshev, Hermite, Laguerre, Jacobi, Legendre polynomials and so on.

None of these have the essential simplicity of the Walsh and Haar functions, the most important examples of non-sinusoidal functions, which form complete sets of orthogonal functions for the Hilbert space $L_2[0,1]$. Having this property, they provide an effective tool in Fourier analysis. With the application of digital techniques and semiconductor technology this kind of complete system of orthogonal functions has been considered and applied [1]. Perhaps this system has other advantages rendering its use more directly applicable to some applications.

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In 1910, Alfred Haar [5] proposed a set of orthogonal functions, taking essentially only two values, such that the formal expansion of a given continuous function in the new functions converges uniformly to the given function. The Walsh functions defined in 1923 by J. L. Walsn [6], formed a complete orthogonal set taking only the values +1 and -1 , and were found to have many properties similar to the sinusoidal series.

From the point of view of approximation theory, it is important to construct a set so that functions in this set cannot be only piece-constant. Schauder basis was obtained by integration of the Haar function. Applying the Schmidt orthonormalization procedure to the Schauder basis, Ciesielski (1968) introduced an orthonormal uniformly bounded sequence of polygonals [3] which was a development of the Franklin orthonormal set discovered in 1928 [4]. The functions in this set are implicit.

In this paper we will give a different class of piecewise linear orthonormal functions U_i which are complete in L_2. They have explicit expressions. We show that any continuous function can be expanded in a (U_i) series which converges uniformly "by group".

2. AN ORTHONORMAL SEQUENCE OF PIECEWISE LINEAR FUNCTIONS

The sequence U, which is the main purpose of this paper to study, consists of the following functions:

$$U_0(x) := 1, \qquad U_1(x) := \sqrt{3}(1-2x), \qquad 0 < x < 1,$$

$$U_2^{(1)}(x) := \begin{cases} \sqrt{3}(1-4x), \\ \sqrt{3}(4x-3), \end{cases} \qquad U_2^{(2)}(x) := \begin{cases} 1 - 6x, & 0 < x < 1/2, \\ 5 - 6x, & 1/2 < x < 1, \end{cases}$$

... ...

$$U_{n+1}^{(2k-1)}(x) := \begin{cases} U_n^{(k)}(2x), & 0 < x < 1/2, \\ (-1)^{k+1}\, U_n^{(k)}(2x-1), & 1/2 < x < 1, \end{cases} \qquad (2.1)$$

$$U_{n+1}^{(2k)}(x) := \begin{cases} U_n^{(k)}(2k), & 0 < x < 1/2, \\ (-1)^{k}\, U_n^{(k)}(2x-1), & 1/2 < x < 1, \end{cases}$$

$$k = 1,2,3,\dots,2^{n-1}, \qquad n = 2,3,\dots,\infty$$

At a point of discontinuity, let these functions be the average of the two one-sided limits.

The first eight of these functions are shown in Fig. 1.

From the definition of the sequence U, it is clear that the function $U_n^{(k)}$ $(n>2)$ is to be used, with the horizontal scale reduced one half and the vertical scale unchanged, to form the functions $U_{n+1}^{(2k-1)}$ and $U_{n+1}^{(2k)}$, the former of which is even and the latter odd with respect to the point $x = 1/2$. All the functions $U_n^{(k)}$ $(n > 2,\ k = 1,2,\dots,2^{n-2})$ have the

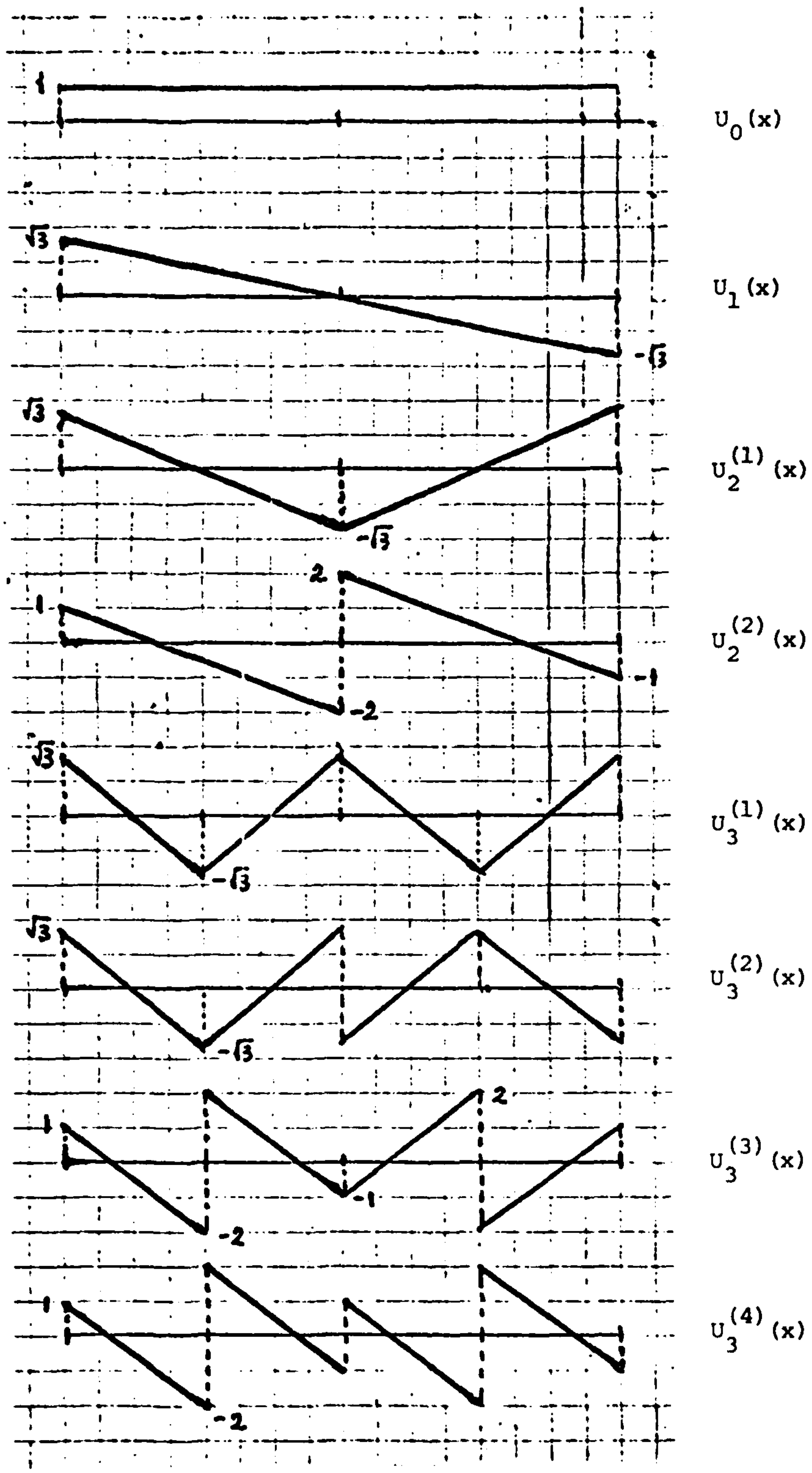

Fig. 1

value $\sqrt{3}$ at $x = 0$ and $(-1)^{k+1}\sqrt{3}$ at $x = 1$, the functions $U_n^{(k)}$ ($n \geq 2$, $k = 2^{n-2}+1, 2^{n-2}+2, \dots, 2^{n-1}$) have the value 1 at $x = 0$ and $(-1)^{k+1}$ at $x = 1$.

Now we consider the orthogonality of the sequence U. At first we establish the following lemma.

<u>LEMMA 1</u> If g satisfies $\int_0^1 g(x)x^i dx = 0$, $i = 0,1$, and g_1 is given by

$$g_1(x) := \begin{cases} g(2x), & 0 \leq x < \tfrac{1}{2}, \\ (-1)^{\ell}\, g(2x-1), & \tfrac{1}{2} \leq x \leq 1, \quad \ell \text{ is an integer} \end{cases}$$

then

$$\int_0^{1/2} g_1(x)x^i dx = \int_{1/2}^1 g_1(x)x^i dx = 0, \; i = 0,1.$$

This lemma follows easily from the change of variable $y = 2x$ and $y = 2x - 1$.

<u>THEOREM 1</u> The sequence of functions $\{U_n^{(i)}\}$ is normal and orthogonal. I.e.

$$\int_0^1 U_n^{(k)}(x)\, U_m^{(j)}(x)dx = \delta_{n,m}\, \delta_{k,j} \tag{2.2}$$

for $n,m = 0,1,2,\dots,$ $k = 1,2,3,\dots,2^{n-1}$ $j = 1,2,3,\dots,2^{m-1}$, with $\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$

<u>PROOF</u> It is easy to check that any two functions $U_n^{(k)}$ are normal and orthogonal if $n \leq 2$.

Assume this fact to hold for $n = 2,3,\dots, N-1$ and $3 \leq m \leq N$. Then

$$\int_0^1 U_N^{(k)}(x)U_m^{(j)}(x)dx = \int_0^{1/2} U_N^{(k)}(x)U_m^{(j)}(x)dx + \int_{1/2}^1 U_N^{(k)}(x)U_m^{(j)}(x)dx \tag{2.3}$$

But, by the method of construction of the sequence U, both integrals in (2.3)

agree, except possibly for sign, with an integral of the form

$$\tfrac{1}{2} \int_0^1 U_{N-1}^{(i)}(y) U_{m-1}^{(\ell)}(y)\,dy \qquad (2.4)$$

after the change of variable $y = 2x$ or $y = 2x-1$ here $i = \left[\frac{k+1}{2}\right]$, $\ell = \left[\frac{j+1}{2}\right]$ respectively.

According to the induction hypothesis the integral (2.4) equals zero if $N \neq m$ or $k \neq j$. We conclude

$$\int_0^1 U_N^{(k)}(x)\, U_m^{(j)}(x)\,dx = 0 \quad \text{for} \quad m \geq 3, \quad m \neq N \quad \text{or} \quad k \neq j \qquad (2.5)$$

Using Lemma 1, it is easy to see that

$$\int_0^1 U_N^{(k)}(x)\, U_0(x)\,dx = 0, \qquad \int_0^1 U_N^{(k)}(x) U_1(x)\,dx = 0$$
$$\int_0^1 U_N^{(k)}(x)\, U_2^{(1)}(x)\,dx = 0, \qquad \int_0^1 U_N^{(k)}(x) U_2^{(2)}(x)\,dx = 0. \qquad (2.6)$$

From (2.5), (2.6) we conclude that

$$\int_0^1 U_N^{(k)}(x)\, U_m^{(j)}(x)\,dx = 0, \quad N \neq m \quad \text{or} \quad k \neq j. \qquad (2.7)$$

By (2.7) and

$$\int_0^1 [U_N^{(k)}(x)]^2 dx = \int_0^{1/2} [U_{N-1}^{(j)}(2x)]^2 dx + \int_{1/2}^1 [U_{N-1}^{(j)}(2x-1)]^2 dx = 1,$$

we confirm that (2.2) holds for $n = N$, and complete the proof.

We denote the collection of all piecewise linear functions with

partition Δ_k by

$$\mathbb{P}_{2,\Delta_k},$$

here Δ_k is the uniform partition on 2^{k-1} intervals. Its dimension is 2^k, since each of its elements consists of 2^{k-1} linear pieces and each piece has 2 freely choosable coefficients.

It is easy to see that

$$U_m^{(k)} \in \mathbb{P}_{2,\Delta_n} \qquad (m = 0,1,\ldots,n, \quad k = 1,2,\ldots,2^{n-1}). \tag{2.8}$$

Hence $M_{2^n} \subseteq \mathbb{P}_{2,\Delta_k}$

$$M_{2^n} := \text{span}\,(U_0,U_1,\ldots,U_n^{(1)},\ldots,U_n^{(2^{n-1})}).$$

But since $\dim M_{2^n} = 2^n$, we get

$$M_{2^n} = \mathbb{P}_{2,\Delta_k} \tag{2.9}$$

From (2.9) we obtain the following theorem.

<u>THEOREM 2</u> If f is a piecewise linear function whose breakpoints can only appear at $\frac{q}{p}$, where q is an integer and p is a power of two, then f can be exactly expressed by finitely terms of the series $\sum \alpha_i U_i$.

3. CONVERGENCE PROPERTIES

Before studying convergence properties we consider the number of sign-changes of functions in the sequences U. At first we define

$$S^-(f) := \sup\{n : \exists\ t_1<t_2<\ldots<t_{n+1}, f(t_i)f(t_{i+1})<0\}$$

to be the number of the sign-changes of f on $[0,1]$.

It is easy to see that

$$S^-(U_0) = 0,\ S^-(U_1) = 1,\ S^-(U_2^{(1)}) = 2 \text{ and } S^-(U_2^{(2)}) = 3.$$

By the method of construction of the sequence U,

$$S^-(U_{n+1}^{(2k-1)}) = 2S^-(U_n^{(k)})$$

and

$$S^-(U_{n+1}^{(2k)}) = 2S^-(U_n^{(k)}) + 1$$

thus,

$$S^-(U_n^{(k)}(x)) = 2^{n-1} + k-1,$$

since this formula holds for $n=2$ and follows for the general case by induction. Hence, each function $U_n^{(k)}$ has one more sign-change than the preceding. Therefore, it is convenient to use the notation $U_0, U_1, U_2, U_3, \ldots$ instead of $U_n^{(k)}$. When we study their sign-changes from now on, we would like to use both $\{U_n^{(k)}\}$ and $\{U_N\}$ freely. Obviously

$$U_n^{(k)} = U_{2^{n-1}+k-1} \quad \text{for } n = 2,3,\ldots, \quad k = 1,2,3,\ldots,2^{n-1}.$$

Thus we get the following theorem.

THEOREM 3 $S^-(U_m) = m$, $m = 0,1,2,3,\ldots$. I.e. $S^-(U_n^{(k)}) = 2^{n-1} + k-1$ for $n = 1,2,3,\ldots$, $k = 1,2,3,\ldots,2^{n-1}$.

Now we begin to study the convergence properties. The Fourier series of

a given function F in terms of the functions U_i is

$$F \sim \sum_{i=0}^{\infty} \alpha_i U_i \tag{3.1}$$

with

$$\alpha_i := (F, U_i) = \int_0^1 F(x)U_i(x)dx. \tag{3.2}$$

Let

$$P_n F := \sum_{i=0}^{n} \alpha_i U_i \tag{3.3}$$

be the n-th partial sum of the series (3.1).

Then $P_n F$ is the best L_2 -approximation to F from $M_n = \text{span } (U_i)_0^n$. Hence it is convergent to F if F is in L_2, since M_n is dense in L_2. Thus we get the following theorem.

<u>THEOREM 4</u> If $F \in L_2[0,1]$, then

$$\lim_{n\to\infty} \|F - P_n F\|_2 = 0 .$$

Next we will prove that $P_{2^n} F$ uniformly approximates $F \in L_\infty$. It is well known [2] that

$$\|F - P_{2^n} F\|_\infty \le (1 + \|P_{2^n}\|) \operatorname{dist}_\infty(F, M_{2^n})$$

and we know

$$\|P_{2^n}\| = \|P_2\| < \infty$$

since least-squares approximation for $M_{2^n} = \mathbb{P}_{2,\Delta_n}$ is local and M_{2^n} is dense in L_∞. Therefore we get the following theorem.

<u>THEOREM 5</u> Let $F \in C[0,1]$, P_{2^n} be L_2-projector onto M_{2^n} on $C[0,1]$, then

$$\lim_{n\to\infty} \|F - P_{2^n} F\|_\infty = 0.$$

4. NOT EVERY CONTINUOUS FUNCTION CAN BE EXPANDED IN TERMS OF THE SEQUENCE U

In this section we prove that there exists a continuous function whose expansion in terms of the U's does not converge at a point of the interval. Our proof rests on a beautiful theorem due to Haar [5].

Suppose $\{\varphi_i\}_{i=1}^{\infty}$ is a complete orthonormal system on $[a,b]$. For a function f, the partial sum f_n of its formal Fourier-series is defined by

$$f_n(s) := \int_a^b K_n(s,t)f(t)dt$$

with

$$K_n(s,t) := \sum_{i=1}^{n} \varphi_i(s)\ \varphi_i(t).$$

THEOREM 6 (Haar [5]) Let $\omega_n(\alpha) := \int_a^b |K_n(\alpha,t)|dt$, here α is an arbitrary point on $[a,b]$. If ω_n is not uniformly bounded for n, then there exists a continuous function $F \in C[a,b]$ such that the series $F_n(s)$ is divergent at the point $s = \alpha$.

In effect, this is one of the first examples of what was called later the principle of uniform boundedness. In our case the kernel is

$$K_n^{(j)}(x,y) := U_0(x)U_0(y) + U_1(x)U_1(y) + \dots + U_n^{(j)}(x)U_n^{(j)}(y).$$

From theorem 5 and 6, for $j = 2^{n-1}$, $\int_0^1 |K_n^{(j)}(\alpha,y)|dy$ is uniformly bounded for n. For general j let

$$K_n^{(j)}(x,y) = K_{n-1}^{(2^{n-2})} + R_n^{(j)}(x,y)$$

I.e.

$$R_n^{(j)}(x,y) := U_n^{(1)}(x)U_n^{(1)}(y) + \dots + U_n^{(j)}(x)U_n^{(j)}(y).$$

Therefore is is sufficient to prove that the integral

$$C_n^{(j)}(\alpha) := \int_0^1 |R_n^{(j)}(\alpha,y)|dy$$

is not uniformly bounded for all n,j.

The following table shows the value of $C_n^{(k)}(0)$ for small value of n and for each value of $k < 2^{n-1}$:

n=2				3/2				
n=3			3/2				3/2	
n=4		3/2		3/2		9/2		3/2
n=5	3/2	3/2	9/2	3/2	21/8	9/2	21/8	3/2

We have the general formulas

$$C_n^{(1)}(0) = C_n^{(2^{n-2})}(0) = 3/2,$$

$$C_n^{(2k)}(0) = C_{n-1}^{(k)}(0),$$

$$C_n^{(2k+1)}(0) = \tfrac{1}{2}\,(C_{n-1}^{(k)}(0) + C_{n-1}^{(k+1)}(0)) + 3/4.$$

So $\int_0^1 |R_n^{(j)}(0,y)|dy$ is not uniformly bounded. We conclude

<u>THEOREM 7</u> There exists a continuous function $f \in C[0,1]$ whose expansion

$$\sum_{i=1}^{n} \int_0^1 f(x)U_i(x)\,dx U_i$$

in terms of $\{U_i\}$ does not converge to $f(x)$ uniformly.

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